

Advanced ML

& AGT

Class 9

zero-Sum Games
+ applications

Zero-Sum Games and

Regret minimization applications

zero sum games ①

max-min CEN, SGIN

P'WIN' ②

minimax work

boosting ③

(מינימיזציה של גיבובים)

Pure Nash Equilibrium

$$a = (a_1, \dots, a_N) \text{ is a profile}$$

$$\forall i \in [N] \quad \forall \bar{a}_i \in A_i$$

$$l_i(a) \leq l_i(a_{-i}, \bar{a}_i)$$

Mixed Nash Equilibrium

$$D = (D_1, \dots, D_N) \text{ is a strategy profile}$$

$$\forall i \in [N] \quad \forall \bar{D}_i \in \Delta(A_i)$$

$$E[l_i(a_1, \dots, a_N)] \leq E[l_i(a_{-i}, \bar{a}_i)]$$

$$a_j \sim D_j$$

$$a_i \sim D_i$$

$$a_j \sim D_j$$

$$\bar{a}_i \sim \bar{D}_i$$

Open 11.8

Suppose N is even

$i \in [N]$ ignore \bar{a}_i and A_i

i ignore \bar{a}_i and D_i

$$l_i(a_1, \dots, a_N) : A_1 \times \dots \times A_N \rightarrow \mathbb{R}$$

$$(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_N) = a_{-i}$$

Take product of all profiles

a_{-i} is given by \bar{a}_{-i}

$$\min_{\bar{a}_i} l_i(a_{-i}, \bar{a}_i)$$

KN & 13

$$\begin{array}{c} b_1 \ b_2 \ b_3 \\ \hline a_1 & \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ -2 & 4 & 1 \end{bmatrix} = M \\ a_2 \\ a_3 \end{array}$$

Game form II

$$\min_i \max_j M[i, j] = 2 \quad (i=1, j=3)$$

$$\max_j \min_i M[i, j] = 0 \quad (i=1, j=2)$$

Game form IIc

$$\min_i \max_j M[i, j] \geq \max_j \min_i M[i, j]$$

2 player Zero-Sum Game

reko type 2 se posen

$$\forall a_1 \in A_1, a_2 \in A_2 \quad l_1(a_1, a_2) + l_2(a_1, a_2) = 0$$

reko type 1b

$$\forall D_1 \subseteq \Delta(A_1), D_2 \subseteq \Delta(A_2) \quad E[l_1(a_1, a_2) + l_2(a_1, a_2)] = 0$$

a*i*nd *j*

reko 3b

rank 3000 2/2 8/2 7/2

minimax loss 23/2 3 rank 1700

minimax loss 23/2 10 rank

רמיינר גאנטן זענער אונדזערן צוינערן

$$v = \min_p \max_q M[p, q] = \min_p \max_j M[p, j]$$

$$u = \max_q \min_p M[p, q] = \max_q \min_i M[i, q]$$

LP -> ויזה אט זונד

min v

$$p^T M \leq v \vec{1}$$

$$Mq \geq u \vec{1}$$

$$p^T \vec{1} = 1$$

$$q^T \vec{1} = 1$$

$$p \geq 0$$

$$q \geq 0$$

u = v \rightarrow סולץ זונד



רמן ניימן

$$p \in \Delta(A_1), q \in \Delta(A_2) \quad M[p, q] = p^T M q$$

$$= \sum_{i,j} p[i] M[i, j] q[j]$$

(Non Neuman) Min Max Goon

$$\min_p \max_q M[p, q] = \max_q \min_p M[p, q]$$

$$Mq = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

inten /prob

absurd /prob

$$P^T M = [1 \ 1 \ 1]$$

(absurd) 1 3 02N /prob 6

! valid 15' 68

$$P = \begin{bmatrix} 6/11 \\ 3/11 \\ 2/11 \end{bmatrix} \quad q = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\begin{array}{c} \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3} \\ 6/11 \quad \left[\begin{array}{ccc} 1 & 0 & 2 \\ 3 & 1 & -1 \\ -2 & 4 & 1 \end{array} \right] \\ 3/11 \\ 2/11 \end{array} = M$$

SCN 6138 0730

: 2021

regret min sick pre' ipne fo

(max ipne) - L_{0n}) min ipne L_{0n} o'300

-2-1 1 ipne L_{mn}² -1 L_{mn}¹: o'300

min ipne L_{mn} 1 300 1 300 1 300

(max ipne) q-1 (min ipne) p

1 300 1 300 1 300

$$U_{\min}^1 \geq L_{\min} / T$$

$$U_{\min}^2 = -U_{\max}^1 \geq L_{\min}^2 / T$$

Minimax Theorem

regret min. sick is o'300

$$U_{\min}^1 = \max_q \min_i M[i, q] \quad ?'300$$

$$U_{\max}^1 = \min_p \max_j M[p, j]$$

U_{min} = U_{max} : Minimax Con

$$U_{\min}^1 \leq U_{\max}^1 : U/c?$$

$$U_{\max}^1 = U_{\min}^1 + \gamma, \quad \gamma > 0 \rightarrow \text{upper bound}$$

2) T-1 sick 200

$$\frac{R(T)}{T} < \frac{\gamma}{2}$$

$L_{ON}/T \leq \delta$ all

$$\frac{L_{ON}}{T} \leq \frac{L'_{min}}{T} + \frac{R(T)}{T} < U'_{min} + \frac{\delta}{2}$$

$$\frac{L_{ON}}{T} \geq \frac{-L^2_{min}}{T} - \frac{R(T)}{T} > U'_{max} - \frac{\delta}{2}$$

$$0 \leq U'_{max} - U'_{min} < \delta$$

μ_{ON}

! note ≈ 0

~~A~~

: regret \Rightarrow $\mathcal{O}(1/\delta)$

$$(1) \text{ pre) } L_{ON} \leq L'_{min} + R(T)$$

$$(2) \text{ pre) } -L_{ON} \leq L^2_{min} + R(T)$$

$$-L^2_{min} - R(T) \leq L_{ON}$$

מבחן רצף עליות -

$1/\sqrt{T}$ פל

הפרון היעדרות יסוד
הפרון היעדרות יסוד

$$\omega_{t+1}(j) = \omega_t(j) \exp(-\eta l_t(j)) \quad : \text{Hedge}$$

$\Theta(1/\sqrt{T})$ רצף

Optimistic Hedge

$$\omega_{t+1}(j) = \omega_t(j) \exp(-\eta(l_t(j) + \underline{l}_t(j) - \bar{l}_t(j-1)))$$

$\Theta(\ln T/T)$ רצף

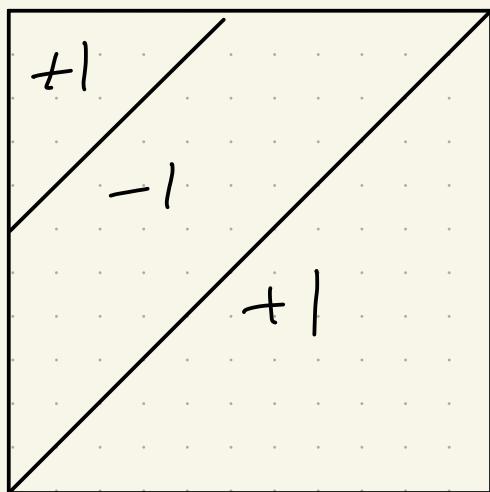
כזה

יתרונות regret גנוי גודל סט -

$$U^* + \frac{R(T)}{\sqrt{T}} \rightarrow \text{lf} \text{ גודל סט}$$

גנוי גודל סט U^*

מינימקסית, איזה סידור מינימקסית



$$\forall D_x \exists y \quad V_I \leq y$$

$$\forall D_y \exists x \quad V_{II} \geq x$$

$$V_{II} \geq 3/7 > 1/3 \geq V_I$$

! גנוי על כל גודל סט גודל סט

regret Min open simple IL

1) fix k regret method

$$\sigma_t = (a_t, b_t) \in A_1 \times A_2$$

2) 3 MN method

$$\bar{\sigma} = \frac{1}{T} \sum_{t=1}^T \sigma_t \in \Delta(A_1 \times A_2)$$

$$L_{\bar{\sigma}} = \frac{1}{T} \text{Cost}(ALG)$$

$$L_{\bar{\sigma}}(i_0) = \frac{1}{T} \text{Cost}(i_0)$$

$$\varepsilon_T = \frac{R(T)}{T} \geq 0$$

$$L_{\bar{\sigma}} \leq L_{\bar{\sigma}}(i_0) + \varepsilon_T$$

Open to new SF IC

Coarse Correlated Equilibrium

Open book rule open

CCE open rule

, σ selected by

(σ prob) "Commit" if no other player

(σ less than prob) "deviate" if

$$L_{\sigma} = E_{(i,j) \sim \sigma} M[i,j] \quad \text{"commit"}$$

$$L_{\sigma}(i_0) = E_{(i,j) \sim \sigma} M[i_0, j] \quad \text{"deviate"}$$

$\forall i_0 \quad L_{\sigma} \leq L_{\sigma}(i_0)$ ok CCE for σ

$\forall i_0 \quad L_{\sigma} \leq L_{\sigma}(i_0) + \varepsilon$ ok ε -CCE for σ

CCE can regret min < 0

Worse Case time

$$\min_i \max_j \text{Time}(A_i, x_j)$$

Non Deterministic Time

$$\max_j \min_i \text{Time}(A_i, x_j)$$

Random worse-case

$$\min_P \max_j R\text{Time}(P, x_j)$$

Average worse-case

$$\max_q \min_i \text{AvgTime}(A_i, q)$$

Deterministic & Randomized Algorithms

⇒ Worst case lock A

⇒ Avg prob lock X

avg prob for A_i $\leq \mu_N \text{Time}(A_i, x_j)$

$\sum_i p_i$ avg prob

A fn P refact

$$R\text{Time}(P, x_j) = \sum_i p_i \text{Time}(A_i, x_j)$$

X fn Q refact

$$\text{AvgTime}(A_i, q) = \sum_j q_j \text{Time}(A_i, x_j)$$

GojwG גודל מינימום תרנגול פ'ס

$$\min_{\bar{P}} \max_j \mathbb{E}[\text{Time}(A_i, x_j)] = \max_q \min_i \mathbb{E}[\text{Time}(A_i, x_j)]$$

\bar{P}, \bar{q} בפ'

$$\min_{\bar{P}} \max_j \mathbb{E}[\text{Time}(A_i, x_j)] \leq \max_j \mathbb{E}[\text{Time}(A_i, x_j)]$$

$$\max_q \min_i \mathbb{E}[\text{Time}(A_i, x_j)] \geq \min_i \mathbb{E}[\text{Time}(A_i, x_j)]$$

$$\max_j \mathbb{E}[\text{Time}(A_i, x_j)] \geq \min_{j \in \bar{q}} \mathbb{E}[\text{Time}(A_i, x_j)]$$

$R\text{Time}(\bar{P}, \bar{x}_j)$

$j \in \bar{q}$

$\text{ArgTime}(A_i, \bar{q})$

V_P, q

(Yao's Lemma) גאנ

$$\max_j R\text{Time}(P, x_j) \geq \min_i \text{ArgTime}(A_i, q)$$

וכוון: (ב) כוון

A_i גאנר : Min-Player

x_j גאנר : Max-Player

$\text{Time}(A_i, x_j)$: פעולה

: Mini-Max גאנר

$$\min_{\bar{P}} \max_q \mathbb{E}[\text{Time}(A_i, x_j)] = \max_{\bar{P}} \min_{\bar{q}} \mathbb{E}[\text{Time}(A_i, x_j)]$$



Best Arm Identification

$(T < \frac{k}{\epsilon})$ j מודפס $k/3$ פעמיים ב- \mathcal{Y}_T

$$\Pr_R[Y_T \neq j | I_j] \geq \frac{1}{4}$$

\mathcal{Y}_T מודפס $n/3$ פעמיים ב- \mathcal{Y}_T

$$M[i,j] = \Pr_R[Y_T \neq j | I_j, A_i]$$

I_j מודפס $n/3$ פעמיים ב- \mathcal{Y}_T

$$E_j[\Pr_R[Y_T \neq j | I_j, A_i]] \geq \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

A_i מודפס $n/3$ פעמיים ב- \mathcal{Y}_T

$$E_{A_i \text{ up}}[E_j \Pr_R[Y_T \neq j | I_j, A_i]] \geq \frac{1}{12}$$

$K/3^2 \cdot (n/3) \geq n/3$ פעמיים ב- \mathcal{Y}_T

לפחות $n/3$ פעמיים ב- \mathcal{Y}_T

מגדיר $\delta = \frac{\epsilon}{(K+1)^2}$
 $\Pr_R[Y_T \neq j | I_j] \geq \frac{1}{4}$
 $\Pr_R[Y_T \neq j | I_j] \geq \frac{1}{4} - \frac{\delta}{(K+1)^2} = \frac{1}{4} - \frac{\epsilon}{(K+1)^2}$

$\Pr_R[Y_T \neq j | I_j] \geq \frac{1}{4} - \frac{\epsilon}{(K+1)^2}$

$\Pr_R[Y_T \neq j | I_j] \geq \frac{1}{4} - \frac{\epsilon}{(K+1)^2} = \frac{1}{4} - \frac{\epsilon}{(n/3)^2} = \frac{1}{4} - \frac{\epsilon}{n^2/9} = \frac{1}{4} - \frac{9\epsilon}{n^2}$

$\Pr_R[Y_T \neq j | I_j] \geq \frac{1}{4} - \frac{9\epsilon}{n^2} \geq \frac{1}{4} - \frac{9\epsilon}{(n/3)^2} = \frac{1}{4} - \frac{9\epsilon}{n^2/9} = \frac{1}{4} - \frac{81\epsilon}{n^2}$

$\Pr_R[Y_T \neq j | I_j] \geq \frac{1}{4} - \frac{81\epsilon}{n^2} \geq \frac{1}{4} - \frac{81\epsilon}{(n/3)^2} = \frac{1}{4} - \frac{243\epsilon}{n^2}$

$\Pr_R[Y_T \neq j | I_j] \geq \frac{1}{4} - \frac{243\epsilon}{n^2} \geq \frac{1}{4} - \frac{243\epsilon}{(n/3)^2} = \frac{1}{4} - \frac{729\epsilon}{n^2}$

$\Pr_R[Y_T \neq j | I_j] \geq \frac{1}{4} - \frac{729\epsilon}{n^2} \geq \frac{1}{4} - \frac{729\epsilon}{(n/3)^2} = \frac{1}{4} - \frac{2187\epsilon}{n^2}$

$\Pr_R[Y_T \neq j | I_j] \geq \frac{1}{4} - \frac{2187\epsilon}{n^2} \geq \frac{1}{4} - \frac{2187\epsilon}{(n/3)^2} = \frac{1}{4} - \frac{6561\epsilon}{n^2}$

$$\max_D \min_h \mathbb{E}_{x \sim D} [M(h, x)] \leq \frac{1}{2} - \gamma \quad : \text{why?}$$

$$V \leq \frac{1}{2} - \gamma \quad : \text{prove by induction}$$

: Min Max Game

$$\exists q \in \Delta(H) \quad \max_{\substack{x \\ h \sim q}} \mathbb{E} M(h, x) \leq \frac{1}{2} - \gamma$$

$$\forall x \quad \mathbb{E}_{h \sim q} [M(h, x)] \leq \frac{1}{2} - \gamma$$

$$G(x) = \sum_h q(h) h(x) \quad \text{by definition}$$

$$\begin{aligned} \forall x \quad |f(x) - G(x)| &= \left| \sum_h q(h) (f(x) - h(x)) \right| \\ &\leq \sum_h q(h) \underbrace{|f(x) - h(x)|}_{M(h, x)} \leq \frac{1}{2} - \gamma < \frac{1}{2} \end{aligned}$$

$$\forall x \quad f(x) = \text{Sign}(G(x) - \frac{1}{2}) \quad \text{proof}$$

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Weak vs Strong Learner: Boosting

$f: X \rightarrow \{0, 1\}$ weak learner : β_M

$H = \{h: X \rightarrow \{0, 1\}\}$ weak learner

γ -WL

$$\forall D \quad \exists h \in H: \Pr_{x \sim D} [h(x) \neq f(x)] \leq \frac{1}{2} - \gamma$$

f is a composition of β_M 's and 3γ : take

weak prob proof

$$M(h, x) = \begin{cases} 0 & f(x) = h(x) \\ 1 & f(x) \neq h(x) \end{cases}$$

γ -WL proof

$$\forall D \quad \exists h \in H \quad \Pr_{x \sim D} [M(h, x)] \leq \frac{1}{2} - \gamma$$

(Hoeffding) ג'ס'ס כוונ'

ר'פונט-ה'ס'ס כוונ'

$$|G_n(x) - G(x)| \leq \delta/2$$

$$n \geq \frac{c}{\delta^2} \log \frac{1}{\varepsilon} \quad \text{ר'פונט}$$

ר'פונט $G_n(x)$ כביצה

שכונה

$$\sum_h q(h)h(x) = G(x) \quad \text{ר'פונט}$$

H בפ'ס'ס כוונ' כ'ז'

G f ג'ס'ס ו'פ'ס'ס זוק

q פונט מושג n מ'ז'

(אחסן פ'ס'ס א'ז')

$$G_n(x) = \sum_{i=1}^n h_i(x)$$

$$E[G_n(x)] = G(x)$$

: 73) ה' 300%

$$l_t(x_i) = \begin{cases} 1 & f(x_i) = h_t(x_i) \\ 0 & \text{otherwise} \end{cases}$$

Regret \geq 'ה' 300%

$\left(\frac{1}{2} + \gamma\right) T \leq 300\% \Rightarrow$

$$\left(\frac{1}{2} + \gamma\right) T \leq 300\% \Rightarrow$$

לכל x_i $\Pr_{h_t \sim \text{alg}}[h_t(x_i) \neq f(x_i)] \leq \frac{1}{2} + \gamma$

לכל x_i $\Pr_{h_t \sim \text{alg}}[h_t(x_i) \neq f(x_i)] \leq \frac{1}{2}$

$$\frac{1}{2} T \geq 300\%$$

$$\left(\frac{1}{2} + \gamma\right) T \leq \text{loss}(\text{RegretAlg}) \leq \frac{T}{2} + 2\sqrt{T \log m}$$

$$T \leq \frac{4 \log m}{\gamma^2} \cdot 100 \text{%"}$$

Regret Min by Boosting 'ה'

x_1, \dots, x_m נספכ' על

: t מגד

ה' מילויו של RegretAlg -

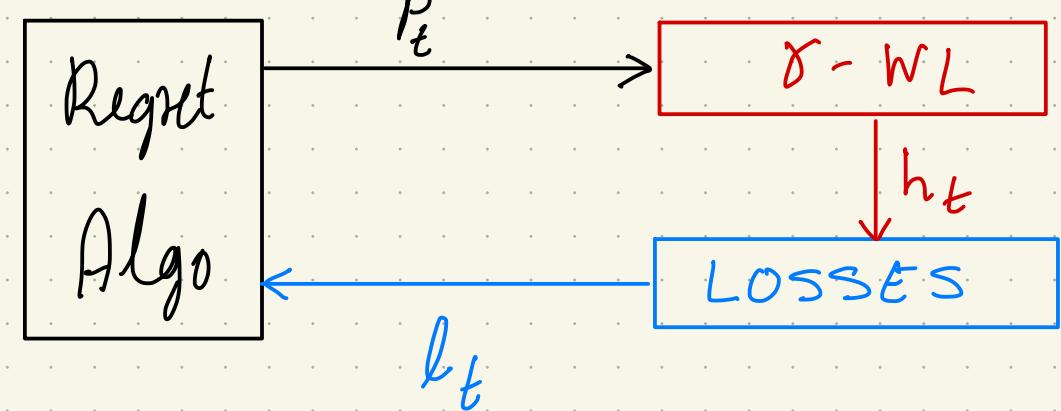
(x_1, \dots, x_m, f_N)

h_t תדריך מוגן -

$$\Pr_{x \sim P_t}[h_t(x) \neq f(x)] \leq \frac{1}{2} - \gamma$$

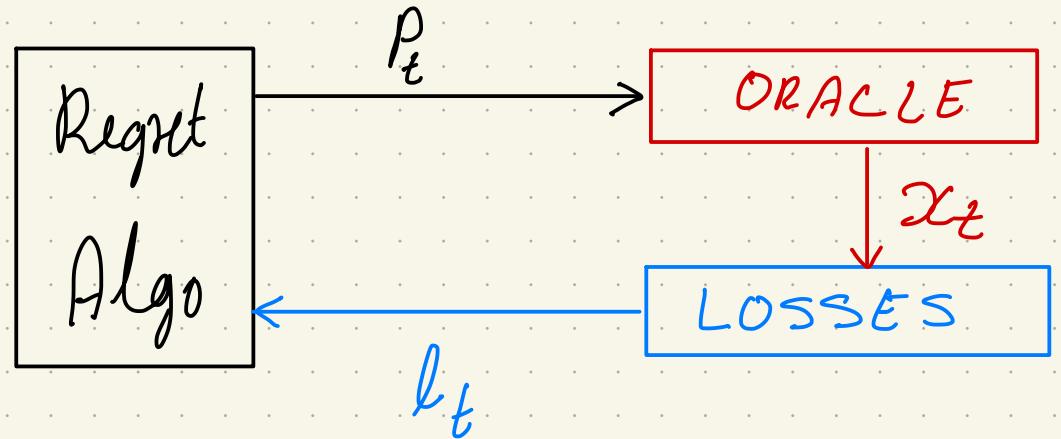
ר'ג'ז'ן T מגד

$$\hat{f}(x) = \text{MAJ}(h_1(x), \dots, h_T(x))$$



רשות הPLAN

: תומס



! הוכיחו שאין ORACLE יפה

: ר'ז סעיף

$$l_t = \frac{1}{\rho} (b - Ax_t) \in [-1, +1]^m$$

$$l_t(i) = \frac{1}{\rho} (b_i - A_i^T x_t) \in [-1, +1]$$

(Packing) Linear Program

$$A \in \mathbb{R}^{m \times n} \quad b \in \mathbb{R}^m \quad K \subseteq \mathbb{R}^n \text{ אוסף}$$

$$Ax \leq b$$

$$Ax \leq b + \epsilon \vec{1} \quad x \in K \cap \{x \mid Ax \leq b\}$$

$$\|Ax - b\|_\infty \leq \rho \quad \text{כפי}$$

ORACLE יפה נוכח

$$p \in \Delta(m) \text{ יסוד}$$

$$x \in K \text{ יפה}$$

$$p^T Ax \leq p^T b \quad \text{כפי}$$

(יהי יפה) יפה יפה

ρ'30077 773C7N

$$0 \leq \frac{1}{T} \sum_{t=1}^T \frac{1}{\rho} (b_i - A_i^T \bar{x}_t) + \frac{R(T)}{T}$$

$$R(T) = 2\sqrt{T \ln m} \quad \text{with } \gamma$$

$$A_i^T \underbrace{\frac{1}{T} \sum_{t=1}^T x_t}_{\bar{x}} \leq b_i + \underbrace{\frac{\rho \cdot 2\sqrt{T \ln m}}{T}}_{\epsilon}$$

$$T = \frac{4\rho^2 \ln m}{\epsilon^2}$$

$$\forall i \quad A_i^T \bar{x} \leq b_i + \epsilon$$

$$A \bar{x} \leq b + \epsilon \vec{1}$$

YDQP

$$T = \frac{4\rho^2 \ln m}{\epsilon^2} \quad \rho' 37$$

$$\bar{x} = \frac{1}{T} \sum_{t=1}^T x_t$$

: ORACLE 773C7N

$$p_t^T l_t = \frac{1}{\rho} (p_t^T b - p_t^T A x_t) \geq 0$$

Regret 10/10 773C7N

$$0 \leq \sum_{t=1}^T p_t^T l_t \leq \sum_{t=1}^T l_t^{(i)} + R(T)$$

Online i in 10

נוילס

ו. כ. נ. ו. כ. נ. ו.

2018/19 & 2009/10

Slivkins, 9 פברואר

ו. כ. נ. ו. כ. נ. ו.

OPEN II.C, OPEN ①

OOD OLD JOE ②

Min-max COEN

regret 'ונתק'

נ. ג. ג. ג. ג. ③

ר. ס. ר. ס. ר. ס. ג. ג. ג. ג. ג. ג. ג. ④

Boosting ⑤

.LP מ. י. ו. ⑥